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MARK BROWN
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by

Mark Brown†
Department of Operations Research
Cornell University
Ithaca, New York

and

Sheldon M. Ross††
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

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Abstract

The theory of cumulative processes, introduced and developed by W. L. Smith, provides a significant generalization of the renewal counting process. We examine the question of extending the Blackwell and key renewal theorems to cumulative processes. For a subclass of cumulative processes, which we call strongly cumulative, the Blackwell and key renewal theorems hold under very general conditions. This class of cumulative processes includes all the standard examples of cumulative processes. We also study processes of the form $Y(t) = \int_0^t V(s)ds$ where V is a regenerative process. Smith has shown that under mild conditions $V(s)$ converges in distribution, say to $V(\infty)$, as $s \to \infty$, and that $Y(t)/t$ converges almost surely and in expectation to κ_1/μ_1 (a constant). Our result is that $\kappa_1/\mu_1 = EV(\infty)$. This holds even if $\lim_{t\to\infty} EV(t)$ does not exist.

Introduction

Cumulative processes were introduced and the theory developed in an elegant paper of W. L. Smith [4]. The essential structure of a cumulative process, $\{Y(t),\ t \geq 0\}$, is given by $Y(t) = \sum_{1}^{N(t)} Y_i + W(t)$, where $N(t)$ is the renewal counting function of an embedded renewal process, $Y_i = Y(T_i) - Y(T_{i-1})$ where T_i is the time of the i^{th} renewal, and $W(t) = Y(t) - Y(T_{N_t})$. If assumptions are introduced which insure that the sequence $\{Y_i\}$ is i.i.d. and $W(t)$ is small compared to $Y(t)$ for t large, then for t large $Y(t)$ behaves like a random sum of i.i.d. random variables, and one can exploit this structure to obtain asymptotic properties of $Y(t)$. Smith derived conditions for almost sure and L_1 convergence of $Y(t)/t$, as well as asymptotic normality of $t^{-1/2}(Y(t) - t/\mu_1)$.

Letting $Y(t) = N(t)$ (in which case $Y_i \equiv 1$ and $W(t) \equiv 0$) it is seen that renewal counting processes are a special case of cumulative processes. One can thus view Smith's results as generalizations to cumulative processes of theorems concerning renewal counting processes. In this context it is of interest to examine the possibility of extending the Blackwell and key renewal theorems, which are results of great importance for renewal counting processes, to cumulative processes. In section 1 we show that for a slightly restricted class of cumulative processes, the above theorems hold if $\tilde{\kappa}_1$ the expected total variation of Y over a renewal cycle, is finite. The restricted class which we consider is characterized by a stronger regeneration property at renewal epochs than that defined by Smith. To our knowledge this class of cumulative processes includes all the standard examples of cumulative processes.

In section 2 we consider cumulative processes of the form $Y(t) = \int_0^t V(s)ds$, where V is a real valued regenerative process. An important result of Smith is that $Y(t)/t$ converges almost surely and in expectation to κ_1/μ_1, where κ_1 is the expected value of the integral of V over a regeneration cycle, and μ_1 is the expected length of a regeneration cycle. Our result is that $\kappa_1/\mu_1 = EV(\infty)$ where $V(\infty)$ has the limiting distribution of $V(t)$.

1. Define $\{X_0, X_1, \ldots\}$ to be a generalized renewal sequence if $\{X_0, X_1, \ldots\}$ are proper $(\Pr(X_i < \infty) = 1)$ non-negative independent random variables, and $\{X_1, X_2, \ldots\}$ are identically distributed. Without loss of generality assume that $\Pr(X_1 = 0) = 0$. Let $\{Y(t),\ t \geq 0\}$ be a real-valued stochastic process. Define $\tilde{Y}(t)$ to be the total variation of Y over $[0,t]$. Assume that $\tilde{Y}(t) < \infty$ a.s. for all t. Define:

(i) $\mu_1 = EX_1$ (possibly $+\infty$)

(ii) $T_0 = 0$, $T_n = \sum_{i=0}^{n-1} X_i$, $n = 1,2,\ldots$

(iii) $N(t) = k$ iff $T_k \leq t < T_{k+1}$

(iv) $m(t) = E(N(t))$

(v) $Y_n = Y(T_{n+1}) - Y(T_n)$, $n = 1,2,\ldots$

(vi) $\tilde{Y}_n = \tilde{Y}(T_{n+1}) - \tilde{Y}(T_n)$, $n = 1,2,\ldots$

(vii) $\tilde{\kappa}_1 = E\tilde{Y}_1$ (possibly $+\infty$), $\kappa_1 = EY_1$ (if it exists).

Smith defines a real valued process $\{Y(t),\ t \geq 0\}$ to be cumulative if it satisfies:

(C1) $\{Y_1, Y_2, \ldots\}$ are i.i.d. random variables.

(C2) $\tilde{Y}(t) < \infty$ a.s. for all t.

(C3) $\{\tilde{Y}_1, \tilde{Y}_2, \ldots\}$ are i.i.d. random variables.

Given a process Y and a generalized renewal sequence $\{X_0, X_1, \ldots\}$, define the process $\{Y'(t),\ t \geq 0\}$ by:

$$\{Y'(t) = Y(X_0 + t),\ t \geq 0\}$$

Similarly define $\{\tilde{Y}'(t),\ t \geq 0\}$. Define a process Y to be strongly cumulative if Y satisfies (C2) and condition (C1') given below:

(C1') $\Pr(Y(t) - Y(T_{N_t}) \in A \mid \text{initial conditions},\ X_0 \leq t,$
$\{Y(x),\ x \leq T_{N_t}\},\ T_{N_t} = s) = \Pr(Y'(t-s) - Y'(0) \in A \mid X_1 > t-s)$ for all $t \geq 0$ and Borel sets A.

Condition (C1') asserts that given T_{N_t} and $X_0 \leq t$, $Y(t) - Y(T_{N_t})$ is conditionally independent of the initial conditions and of the history of the processes until T_{N_t}. Moreover, this conditional distribution depends on t and T_{N_t} only through $t - T_{N_t}$. The initial conditions refer to the joint distribution of X_0 and $\{Y(t),\ t \leq X_0\}$. Condition (C1') is closely related to Smith's definition of a regenerative process, [4] p. 13.

It follows immediately that (C1') implies (C1) and (C3). To see that (C1) and (C3) do not imply (C1'), let $X_0 = 0$, $X_i \equiv 1$ $i > 0$, and choose f_1 and f_2 on $[0,1]$ such that $f_1 \neq f_2$, f_1 and f_2 both have total variation 1 on $[0,1]$ and $f_i(0) = f_i(1) = 0$, $i = 1,2$.

$$\text{Define} \quad Y(t) = \begin{cases} f_1(\alpha), & \text{if } [t] \text{ is even and } t - [t] = \alpha \\ f_2(\alpha), & \text{if } [t] \text{ is odd and } t - [t] = \alpha \end{cases} .$$

Then Y is cumulative but not strongly cumulative. It was this type of example which led us to define strongly cumulative processes, since in dealing with asymptotic local behavior of $Y(t)$, it is convenient to have the entire probabilistic structure of Y the same for each renewal interval, rather than merely having increments $(Y(T_n) - Y(T_{n-1}), \tilde{Y}(T_n) - \tilde{Y}(T_{n-1}))$ identically distributed.

Since we will be generalizing and using the Blackwell and key renewal theorems, we state them here for convenience. The Blackwell and key renewal theorems are equivalent ([2], p. 349).

<u>Lemma 1</u>. Blackwell's theorem. [2] p. 347. Let $\{X_0, X_1, \ldots\}$ be a generalized renewal sequence, with F non-lattice. Then $\lim_{t\to\infty} m(t+h) - m(t) = h/\mu_1$ for all h (if $\mu_1 = \infty$, $1/\mu_1 = 0$).

<u>Lemma 2</u>. Key renewal theorem (Smith, Feller), [2], p. 349. Let $\{X_0, X_1, \ldots\}$ be a generalized renewal sequence with F non-lattice, and z directly Riemann integrable (for all $\varepsilon > 0$ $\exists$ $h > 0$ $\ni$ $h < h' \Rightarrow h \sum_n (\sup_{x\in[(n-1)h,nh)} z(x) - \inf_{x\in[(n-1)h,nh)} z(x) < \varepsilon$). Then $\lim_{t\to\infty} \int_0^t z(t-x)m(dx) = \frac{1}{\mu_1}\int_0^\infty z(x)dx$.

We note that a function on $[0,\infty)$ which is of bounded variation and integrable is directly Riemann integrable. This follows easily from example (b) p. 349 of Feller [2].

__Theorem 1.__ Blackwell theorem for strongly cumulative processes. Suppose $X_0 = 0$, and $\{X_1,X_2,\ldots\}$ is a renewal sequence. Let $\{Y(t),\ t \geq 0\}$ be strongly cumulative relative to $\{X_1,X_2,\ldots\}$. Then if F is non-lattice and $\breve{\kappa}_1 < \infty$ then $\lim_{t\to\infty} E(Y(t+h) - Y(t)) = h\,\frac{\kappa_1}{\mu_1}$ for all h.

__Proof.__ If we define $Y^*(t) = Y(t) - Y(0)$ for $t \geq 0$, then Y^* is strongly cumulative and $Y^*(t+h) - Y^*(t) = Y(t+h) - Y(t)$. We therefore can assume that $Y(0) = 0$ without loss of generality. Note that $\tilde{\kappa}_1 < \infty \Rightarrow \kappa_1$ exists. Also $E|Y(t)| \leq E\tilde{Y}(t) \leq E\tilde{Y}(T_{N_t+1}) = \kappa_1(m(t) + 1)$ by Wald's identity (N_t+1 is a stopping time), and therefore $\tilde{\kappa}_1 < \infty \Rightarrow EY(t)$ exists for all t. Define $R(t) = Y(T_{N_t+1}) - Y(t)$, for $t \geq 0$. Similarly define R' for the Y' process. Now:

$$\begin{aligned} E(Y(t+h) - Y(t)) &= E\Big(\sum_{1}^{N_{t+h}+1} Y_i - \sum_{1}^{N_t+1} Y_i + (R(t+h) - R(t))\Big) \\ &= \kappa_1(m(t+h) - m(t)) + E(R(t+h) - R(t))\ . \end{aligned} \tag{1}$$

By lemma 1, $\lim_{t\to\infty} (m(t+h) - m(t)) = h/\mu_1$, and thus theorem 1 will be proved upon showing that $\lim_{t\to\infty} E(R(t+h) - R(t)) = 0$.

All integrals below, unless otherwise specified, are taken over closed intervals. Now:

$$\text{(2)} \qquad ER(t) = \int_0^t m(dx)(1 - F(t-x))E(R'(t-x)|X_1 > t - x)$$

$$= \text{(3)} \qquad \int_0^t m(dx) \int_{(t-x)^+}^{\infty} E(R'(t-x)|X_1 = s)F(ds)$$

Expression (2) is obtained by conditioning on T_{N_t}. Using expression (3) it follows that:

$$\text{(4)} \qquad E(R(t) - R(t+h)) = a(t) + b(t) - c(t) - d(t) + e(t)$$

where

$$\text{(5)} \qquad a(t) = \int_0^t m(dx) \int_{(t-x+h)^+}^{\infty} E(Y'(t-x+h) - Y'(t-x)|X_1 = s)F(ds)$$

$$\text{(6)} \qquad b(t) = \int_0^t m(dx) \int_{(t-x)^+}^{(t-x+h)} E(Y_1'|X_1 = s)F(ds)$$

$$\text{(7)} \qquad c(t) = \int_0^t m(dx) \int_{(t-x)^+}^{(t-x+h)} E(Y'(t-x)|X_1 = s)F(ds)$$

$$\text{(8)} \qquad d(t) = \int_{t^+}^{t+h} m(dx) \int_{(t-x+h)^+}^{\infty} E(Y_1'|X_1 = s)F(ds)$$

$$\text{(9)} \qquad e(t) = \int_{t^+}^{t+h} m(dx) \int_{(t-x+h)^+}^{\infty} E(Y'(t-x+h)|X_1 = s)F(ds)$$

We will show that each of a, b, c, d, e converges as $t \to \infty$, and evaluate the limits.

(i) It follows from (5) that $a(t) = \int_0^t \alpha(t-x)m(dx)$ where

$\alpha(x) = \int_{(x+h)^+}^{\infty} E(Y'_{x+h} - Y'_x | X_1 = s)F(ds)$.

Define $Z(x) = \begin{cases} Y'(x), & x < \max(X_1 - h, 0) \\ Y'(X_1), & x \geq \max(X_1 - h, 0) \end{cases}$. Then $\alpha(x) = E(Z(x+h) - Z(x))$

and the total variation of Z on $[0,\infty)$ does not exceed $\tilde{Y}'_1$. Let π be a partition of $[0,\infty)$. Now:

$$\sum_{x_i \epsilon \pi} |\alpha(x_i) - \alpha(x_{i-1})| \leq \sum_{x_i \epsilon \pi} E|Z(x_i) - Z(x_{i-1}) + \sum_{x_i + h \epsilon \pi + h} E|Z(x_i + h) - Z(x_{i-1} + h)|$$

$$\leq 2\tilde{Y}_1 < \infty .$$

Therefore α is of bounded variation. Moreover $\int_0^\infty |\alpha(x)|dx = \int_0^\infty E|Z(x+h) - Z(x)|dx \leq 2 \int_0^\infty E|Z(x)|dx \leq 2\tilde{\kappa}_1$. Therefore α is of bounded variation and integrable, and thus directly Riemann integrable. Applying lemma (2) we obtain:

$$\text{(10)} \qquad \lim_{t\to\infty} a(t) = \frac{1}{\mu_1} \int_{x=0}^{\infty} \int_{s=(x+h)^+}^{\infty} E(Y'(x+h) - Y'(x)|X_1 = s)F(ds) = a$$

(ii) It follows from (6) that $b(t) = \int_0^t \beta(t-x)m(dx)$ where

$$\beta(x) = \int_{x^+}^{x+h} E(Y'_1|X_1 = s)F(ds) .$$

Now letting $G(x) = \int_0^x E(Y_1'|X_1 = s)F(ds)$ we see that G is the c.d.f. of a finite signed measure (since $E|Y_1'| < \infty$) and is therefore of bounded variation. Since $\beta(x) = G(x+h) - G(x)$, β is also of bounded variation. Also $\int_0^\infty |\beta(x)|dx \leq \int_0^\infty \int_x^{x+h} E(|Y_1'|\,|\,X_1 = s)F(ds) \leq \int_0^h sE(|Y_1'|\,|X_1 = s)F(ds) +$ $\int_h^\infty hE(|Y_1'|\,|X_1 = s)F(ds) \leq h\tilde{\kappa}_1$. Therefore β is directly Riemann integrable. By lemma (2):

$$\text{(11)} \qquad \lim_{t\to\infty} b(t) = \frac{1}{\mu_1}\int_0^h sE(Y_1'|X_1 = s)F(ds) + \frac{h}{\mu_1}\int_{h^+}^\infty E(Y_1'|X_1 = s)F(ds)$$

$$= b$$

(iii) It follows from (7) that $c(t) = \int_0^t \gamma(t-x)m(dx)$ where $\gamma(x) = \int_{x^+}^{x+h} E(Y'(x)|X_1 = s)F(ds)$.

Define: $L(x) = \begin{cases} Y'(x), & \max(0, X_1 - h) \leq x < X_1 \\ 0 \;, & x < \max(0, X_1 - h),\ x \geq X_1 \end{cases}$. Then $\gamma(x) = EL(x)$, and the total variation of W on $[0,\infty)$ does not exceed $2\tilde{Y}_1'$. Therefore γ is of bounded variation. It is easily shown that γ is integrable. Again, then, by lemma 2:

$$\text{(12)} \qquad \lim_{t\to\infty} c(t) = \frac{1}{\mu_1}\int_0^\infty dx \int_{x^+}^{x+h} E(Y'(x)|X_1 = s)F(ds) = c \;.$$

(iv) It follows from (8) that $d(t) = \int_0^{t+h} \delta(t+h-x)m(dx)$ where

$$\delta(x) = [\int_{x^+}^{\infty} E(Y_1'|X_1 = s)F(ds)]I_{[0,h)}^{(x)} \quad \text{where} \quad I_{[0,h)}^{(x)} = \begin{cases} 1 & 0 \le x < h \\ 0 & x > h, \quad x < 0 . \end{cases}$$

Now $\delta(x) = (1 - G(x))I_{[0,h)}^{(x)}$, the product of two functions of bounded variation. Therefore δ is of bounded variation, and is bounded and vanishes outside a bounded interval and therefore integrable. Using lemma (2) we thus obtain:

$$(13) \quad \lim_{t\to\infty} d(t) = \frac{1}{\mu_1} \int_{s=0}^{h} sE(Y_1'|X_1 = s)F(ds) + \frac{h}{\mu_1}\int_{h^+}^{\infty} E(Y_1'|X_1 = s)F(ds) = d$$

(v) From (8) $e(t) = \int_0^{t+h} m(dx)\varepsilon(t+h-x)$ where

$\varepsilon(x) = [\int_{x^+}^{\infty} E(Y'(x)|X_1 = s)F(ds)]I_{[0,h)}^{(x)}$. By now familiar arguments e is directly Riemann integrable and:

$$(14) \qquad \lim_{t\to\infty} e(t) = \frac{1}{\mu_1}\int_0^{h^-} dx \int_{x^+}^{\infty} E(Y'(x)|X_1 = s)F(ds) = e$$

Note that $a - c + e = 0$ and $b = d$ so that $E(R(t) - R(t+h)) \to 0$ as $t \to \infty$. This proves the result.

<u>Corollary 1.</u> Let $\{X_0, X_1, \ldots\}$ be a generalized renewal sequence, F_0 the c.d.f. of X_0, and F the c.d.f. of X_1. Assume that $\{Y(t), t \ge 0\}$ is strongly cumulative with respect to $\{X_0, X_1, \ldots\}$. Then if F is non-lattice,

$E\tilde{Y}_0 < \infty$, and $\tilde{\kappa}_1 < \infty$ then:

$$\lim_{t\to\infty} E(Y(t+h) - Y(t)) = h\,\frac{\kappa_1}{\mu_1}$$

Proof. Let $h_t(x) = E(Y'(y-x+h) - Y'(t-x))$. Then:

$$(15)\qquad E(Y(t+h) - Y(t)) = \int_0^t h_t(x)F_0(dx) + \int_{t^+}^{\infty} E(Y(t+h) - Y(t)|X_0 = x)F_0(dx)$$

Now $h_t(x) \to h\,\frac{\kappa_1}{\mu_1}$ as $t \to \infty$ for all x (lemma 1), $h_t(x) \geq 0$ for all t, x, and $\sup_{t,x} h_t(x) \leq E\tilde{Y}'(T) + h\,\frac{\kappa_1}{\mu_1} + 1$ where T is chosen so that $x > T \Rightarrow |E(Y'(x+h) - Y'(x)) - h\,\frac{\kappa_1}{\mu_1}| < 1$. Therefore by the dominated convergence theorem $\int_0^t h_t(x)F_0(dx) \to h\,\frac{\kappa_1}{\mu_1}$ as $t \to \infty$. Now

$$\int_{t^+}^{\infty} E(Y(t+h) - Y(t)|X_0 = x)F_0(dx) = \int_{(t+h)^+}^{\infty} E(Y(t+h) - Y(t)|X_0 = x)F_0(dx)$$

$$+ \int_{t^+}^{t+h} E(Y(t+h)|X_0 = x)F_0(dx) - \int_{t^+}^{t+h} E(Y(t)|X_0 = x)F_0(dx)$$

and: $$\left|\int_{(t+h)^+}^{\infty} E(Y(t+h) - Y(t)|X_0 = x)F_0(dx)\right| \leq \int_{(t+h)^+}^{\infty} E(\tilde{Y}_0|X_0 = x)F_0(dx) \to 0$$

as $t \to \infty$ since $E\tilde{Y}_0 < \infty$.

$$\left|\int_{t^+}^{t+h} E(Y(t+h)|X_0 = x)F_0(dx)\right| \leq E\tilde{Y}'(h)(F_0(t+h) - F_0(t)) \to 0$$

as $t \to \infty$.

$$\left|\int_{t^+}^{t+h} E(Y(t)|X_0 = x)F_0(dx)\right| \leq \int_{t^+}^{t+h} E(\tilde{Y}_0|X_0 = x)F_0(dx) \to 0 \text{ as } t \to \infty.$$

In Corollary 1 we impose the condition $E\tilde{Y}_0 < \infty$. An uncontrived example which shows that this condition is not necessary is obtained by defining $\{Y(t) = t,\ t \geq 0\}$. In this case $\tilde{Y}_0 = X_0$. Choose X_1 such that $E|X_1| < \infty$, $EX_1^2 = \infty$, and let X_0 be distributed according to the equilibrium excess life distribution of X_1 $(F_0(x) = \frac{1}{\mu_1}\int_0^x (1 - F(s))ds)$. Then $E\tilde{Y}_0 = \infty$ but $E(Y(t+h) - Y(t)) \equiv h = h\,\frac{\kappa_1}{\mu_1}$.

In the general strongly cumulative case with F non-lattice and $\tilde{\kappa}_1 < \infty$, the conclusion of theorem 1 need not hold. For example, define $\{Y(t) = At,\ t \leq X_0\}$ for $t \leq X_0$, where A is a random variable with $E|A| = \infty$, and choose X_0 such that $Pr(X_0 > x) > 0$ for all x. Distribute $\{Y(t) - Y(X_0),\ t > X_0\}$ as any strongly cumulative process $\{Y'(t),\ t \geq 0\}$ with $X_0' = 0$. Then the process $\{Y(t),\ t \geq 0\}$ will be strongly cumulative, but $EY(t)$ will not exist for any t.

For an important class of strongly cumulative processes, the conclusion of theorem 1 holds without regard to whether or not $E\tilde{Y}_0 < \infty$. This class is obtained by treating the point 0 as a random point in a renewal interval. We assume that the last renewal at or prior to 0 occurred at the point $-T$, where T is a non-negative random variable with c.d.f. G. Moreover we assume that:

(C4) $\{Y(-T+t) - Y(-T),\ t \geq 0\}$ and $\{Y'(t) - Y'(0),\ t \geq 0\}$

are identically distributed.

Since, $E(Y(t+h) - Y(t)) = \int_0^\infty G(dx)E(Y'(t+x+h) - Y'(t+x))$, if F is non-lattice and $\tilde{\kappa}_1 < \infty$, we can apply the dominated convergence argument of Corollary 1 to obtain the conclusion. Thus:

Corollary 2. If $\{Y(t),\ t \geq 0\}$ is strongly cumulative and satisfies condition (C4), then if F is non-lattice and $\kappa_1 < \infty$, then

$$\lim_{t\to\infty} E(Y(t+h) - Y(t)) = h\,\frac{\kappa_1}{\mu_1}.$$

As is shown in Feller ([2], p. 349) lemmas 1 and 2 are equivalent. Moreover, Feller's argument carries over verbatim to the cumulative case. Thus:

Corollary 3. Under the conditions in theorem 1 or of Corollaries 1 or 2, if z is directly Riemann integrable then $\lim_{t\to\infty} \int_0^t z(t-x)m(dx) = \frac{\kappa_1}{\mu_1} \int_0^\infty z(x)dx.$ Moreover, for strongly cumulative processes the Blackwell and key renewal theorems are equivalent.

2. Define G to be the class of all probability distributions on $[0,\infty)$ having the property that for some k, the kth convolution of F with itself has an absolutely continuous component.

Let $\{X_0,X_1,\ldots\}$ be a generalized renewal sequence and $\{V(t),\ t \geq 0\}$ an (X,A) valued stochastic process, where X is a set and A a σ-algebra of subsets of X. Define $\{V'(t) = V(X_0+t),\ t \geq 0\}$. A process V is defined by Smith [4], p. 13, to be regenerative if $\Pr(V(t) \in A \mid \text{initial conditions},\ X_0 \leq t,\ \{V(x),\ x \leq T_{N_t}\},\ T_{N_t} = s) = \Pr(V'(t-s) \in A \mid X_1 > t-s)$ for all

$t \geq 0$, $A \in \mathcal{A}$. Smith has shown, [3], p. 259, that if V is regenerative, $F \in \mathcal{G}$, and $\mu_1 < \infty$, then

$$(16) \qquad \lim_{t\to\infty} \Pr(V(t) \in A) = \frac{1}{\mu_1}\int_0^\infty \Pr(V'(t) \in A,\ X_1 > t)dt = \mu_\infty(A)$$

for all $A \in \mathcal{A}$.

It follows from Fubini's theorem that μ_∞ is a probability measure on $(X,\mathcal{A})$.

Suppose that V is a real-valued regenerative process with a measurable modification, satisfying $\int_0^t E|V(s)|ds < \infty$ for all t. Define $\{Y(t) = \int_0^t V(s)ds,\ t \geq 0\}$. It follows easily that Y is strongly cumulative. Smith, [3], p. 262, has shown that if $\tilde{\kappa}_1 = E\int_0^{X_1}|V'(s)|ds < \infty$, then $Y(t)/t = \frac{1}{t}\int_0^t V(s)ds \to \frac{\kappa_1}{\mu_1}$ a.s. in expectation as $t \to \infty$. In view of this result and lemma 3 it is natural to conjecture that $\kappa_1/\mu_1 = EV(\infty) = \int_{-\infty}^{\infty} x\mu_\infty(dx)$. We will prove this conjecture.

It will be convenient to convert the imbedded renewal process into a stationary renewal process. This is done by inserting a renewal to the left of 0, its distance from 0 coinciding with the asymptotic distribution of $t-T_{N_t}$. Formally, we let $\{X_i^*,\ i = 0,\pm1,\ldots\}$ be a doubly infinite sequence of i.i.d. random variables distributed as X_1. Define $T_n^* = X_0^* + \sum_1^n X_i^*$ for $n \geq 0$, $T_n^* = X_0^* - \sum_1^{-n} X_{-i}^*$ for $n < 0$. Then $\{T_n^*,\ n = 0,\pm1,\ldots\}$ generates a stationary renewal process on $(-\infty,\infty)$ (see [1], p. 162). Define T^0 as the first T_n^* point to the left of 0. Since the process $\{t-T_{N_t}^*,\ t \geq 0\}$ is strictly stationary and regenerative, it follows that the distribution of $-T_0$ coincides with the limiting distribution of

$t - T_{N_t}$. By lemma (3)

$$(17) \qquad Pr(-T_0 \le x) = \frac{1}{\mu_1} \int_0^x (1 - F(s))ds$$

We introduce a process $\{V^*(t),\ t \ge T^0\}$, defined to be identically distributed as $\{V'(t - T^0),\ t \ge 0\}$. Then V^* is a strictly stationary regenerative process.

<u>Theorem 2.</u> Let $\{V(t),\ t \ge 0\}$ be a real-valued regenerative process possessing a measurable modification and such that $F \in G$, $\mu_1 < \infty$, and $\int_0^t E|V(s)|ds < \infty$ for all t. Then $\tilde{\kappa}_1 = E \int_{X_0}^{X_0+X_1} |V(s)|ds < \infty$ iff $E|V(\infty)| < \infty$, and $\tilde{\kappa}_1 < \infty$ implies $\kappa_1/\mu_1 = EV(\infty)$ and $\frac{1}{t}\int_0^t V(s)ds \to EV(\infty)$ a.s. and in expectation as $t \to \infty$.

<u>Proof.</u>

$$(18) \qquad Pr(V^*(0) \in A) = \frac{1}{\mu_1} \int_{t=0}^{\infty} Pr(V^*(0) \in A | T^0 = -t)(1 - F(t))dt$$

and

$$(19) \qquad Pr(V'(t) \in A,\ X_1 > t) = Pr(V'(t) \in A | X_1 > t)(1 - F(t))$$

$$= Pr(V^*(0) \in A | T^0 = -t)(1 - F(t))$$

thus from (16), (18) and (19):

$$(20) \qquad Pr(V^*(0) \in A) = \mu_\infty(A) \quad \text{for all Borel sets .}$$

Define: $$W(t) = \begin{cases} V'(t), & t \le X_1' \\ 0, & t > X_1' \end{cases}$$

Then $\int_0^t |V'(s)|ds = \int_0^\infty |W(s)|ds$, possibly both $+\infty$. Now:

$$E\int_0^\infty |W(s)|ds = \int_0^\infty E|W(s)|ds = \int_0^\infty E(|V'(s)|\,\big|\,X_1' > s)(1 - F(s))ds$$
$$= \int_0^\infty E(|V^*(0)|\,\big|\,T^0 = -s)(1 - F(s))ds = \mu_1 E|V^*(0)| = \mu_1 E|V(\infty)|$$

with all above equalities between integrals holding when the value of any one is $+\infty$. Thus $\tilde{\kappa}_1 < \infty$ iff $E|V(\infty)| < \infty$. If $\tilde{\kappa}_1 < \infty$ then all the above equalities hold with $|W|$, $|V'|$, $|V|$ and $|V^*|$ replaced by W, V', V and V^*. Thus $\kappa_1/\mu_1 = EV(\infty)$. By this equality and lemma 3, $\frac{1}{t}\int_0^t V(s)ds \to EV(\infty)$ a.s. and in expectation as $t \to \infty$.

3. Comments and Additions

(i) A shot noise process, is a process of the form $X(t) = \sum_{T_i \le t} g(t - T_i)$, where g is a real-valued function and $\langle T_i,\ i = 0,1,\ldots\rangle$ a generalized renewal sequence. One can rewrite $X(t) = \int_0^t g(t-s)N(ds)$, which suggests replacing the renewal counting process N, by a strongly cumulative process Y, with $X(t) = \int_0^t g(t-s)Y(ds)$. Then, if Y satisfies the conditions in either corollary 1 or 2, F is non-lattice and g directly Riemann integrable, then by Corollary 3, $EX(t) \to \frac{\kappa_1}{\mu_1}\int_0^\infty g(x)dx$. If $g(x) = e^{-\beta x}$, $\beta > 0$, then $EX(t) \to \frac{\kappa_1}{\beta\mu_1}$. The exponential form for g is usually called a discount factor and is used in models where $Y(t)$ represents the accumulation of capital until time t, and the real value of capital is assumed to decrease exponentially.

(ii) Consider a positive recurrent irreducible Markov renewal process X, with states $0,1,\ldots$. We let the imbedded renewal process consist of the epochs t, for which $X(t^-) \neq 0$, $X(t) = 0$. Define $V(t) = \begin{cases} 0 & X(t) \neq j \\ 1 & X(t) = j \end{cases}$.

Then V is a regenerative process, and thus if $F \in G$ $Y(t)/t = \frac{1}{t}\int_0^t V(s)ds$, the proportion of time spent in state j in $[0,t]$, converges a.s. and in expectation (theorem 2) to $Pr(V(\infty) = j)$. By theorem 1, $E\int_t^{t+h} V(s)ds \to hPr(V(\infty) = j)$. Theorem 1 holds with F non-lattice and in this case theorem 2 will also hold with F non-lattice.

(iii) In the following example, the process Y satisfies (C1'), does not satisfy (C2), and the Blackwell conclusion does not hold. Define $Y(t) = \begin{cases} 0 & t \text{ rational} \\ 1 & t \text{ irrational} \end{cases}$. Let $X_0 = 0$, and let $X_1, X_2, \ldots$ have a non-lattice distribution concentrated on the rationals. Then $\overline{\lim}\ E(Y(t + \sqrt{3}) - Y(t)) = 1$, $\underline{\lim}\ E(Y(t + \sqrt{3}) - Y(t)) = -1$.

(iv) The Wiener process satisfies (C1') (with respect to arbitrary renewal sequences independent of the process itself), does not satisfy (C2), but $E(Y(t+h) - Y(t) \equiv 0$, so the Blackwell conclusion holds.

In general, consider a process $\{Y(t), t \geq 0\}$ satisfying (C1') with κ_1 existing. Then it follows from (1), that a necessary and sufficient condition for $\lim_{t\to\infty} E(Y(t+h) - Y(t)) = \frac{\kappa_1}{\mu_1} h$, is that $E(R(t+h) - R(t)) \to 0$ as $t \to \infty$. A sufficient condition is that $\lim_{t\to\infty} ER(t)$ exist.

(v) Consider a $G \cdot I \cdot |G|s$ queue, $1 \leq s \leq \infty$, with or without batch arrivals. A renewal epoch occurs when a customer initiates a busy period. For particular queuing models conditions are available under which $\mu_1 < \infty$. For example, in a $G \cdot I \cdot |G|s$ queue $1 \leq s < \infty$, with single arrivals, $\mu_G/\mu_A < 1$ (expected service time/expected interarrival time) is sufficient. If either the interarrival time distribution or service time distribution is non-lattice (belongs to $\mathcal{G}$) then F is non-lattice (belongs to $\mathcal{G}$). Some interesting regenerative and cumulative processes are:

(1) $V_1(t)$ = number of customers in queue at time t. $Y_1(t) = \int_0^t V_1(s)ds$ = total waiting time of customers in queue during $[0,t]$.

(2) $V_2(t) = \begin{cases} 1 & \text{if } j \text{ customers in queue at time } t. \\ 0 & \neq j \end{cases}$

$Y_2(t)$ = proportion of time in $[0,t]$ with j customers in the queue.

(3) In (1) and (2) replace customers in queue, by customers receiving service.

(vi) In Theorem (2), $F \in \mathcal{G}$ can be replaced by F non-lattice, as long as the class of sets A for which $\phi_A(t) = Pr(V'(t) \in A, X_1 > t)$, is of bounded variation on finite intervals, generates the Borel sets. This follows from Smith [3], p. 259.

(vii) Let S be a regenerative process with state space $(X,\mathcal{A})$ and possessing a modification which is jointly measurable as a map from $(\Omega,\mathcal{C})\times(R,\mathcal{B})$ to $(X,\mathcal{A})$.

Here (Ω,C,P) is the probability space on which each $S(t)$ is defined, $R = [0,\infty)$, $\mathcal{B}$ the Borel sets on $[0,\infty)$. If R is a real-valued Borel measurable function, then $\{V(t) = R(S(t)),\ t \geq 0\}$ is a real-valued measurable regenerative process. If $\mu_1 < \infty$, $F \in G$, then since $Pr(S(t) \in A) \to Pr(S(\infty) \in A)$ for all $A \in A$, it follows that $Pr(R(S(t)) \in B) \to \mu_\infty(R^{-1}(B))$, for all Borel sets. Thus if $E|R(S(\infty))| < \infty$, then if follows from theorem 2 that:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(S(x))dx = ER(S(\infty))$$

a.s. and in expectation.

(viii) If $EV(t)$ converges then $EV(\infty)$ must be its limit, since $EV(\infty) = \lim_{t\to\infty} \frac{1}{t} \int_0^t E(V(s))ds$. In this case theorem 2 is trivial. However, $E(V(t))$ may not converge and theorem 3 may still hold. For example, let $X_0 = 0$, and let $F \in G$ have an atom at 1. Choose a regenerative process V so that $E(V(t)|t - T_{N_t} = 1/2) = \infty$, $E(V(t)|t - T_{N_t} \neq 1/2) = 0$. Then clearly $EV(n + 1/2) = \infty$ for all n, but $EV(\infty) = 0$.

When $X_0 = 0$, a necessary and sufficient condition for convergence of $EV(t)$ to $EV(\infty)$ is uniform integrability of $g(\cdot) = E(V(t)|t - T_{N_t} = \cdot)$ with respect to the family $\{F_t,\ t \geq 0\}$, where $F_t(x) = Pr(t - T_{N_t} \leq x) =$

$$= \begin{cases} 1 - \int_0^{(t-x)^-} m(ds)(1 - F(t-s)), & x < t \\ 1 & x \geq t \end{cases}$$

REFERENCES

[1] Beutler, F. J. and Leneman, O. A. Z., The theory of stationary point processes, Acta Mathematica, (1966), pp. 159-197.

[2] Feller, William, An Introduction to Probability Theory and Its Applications, Volume II, John Wiley and Sons, New York, 1966.

[3] Smith, W. L., Renewal Theory and its ramifications, J. R. Statist. Soc. B, 20 (1958), pp. 243-302.

[4] Smith, W. L., Regenerative stochastic processes, Proc. Roy. Soc. A, 232 (1955), pp. 6-31.